Problem solving for Midterm Test 2

Stochastics

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Problem 1

Every morning, Jack buys a sandwich for breakfast. He always buys either an egg sandwich or a cheese sandwich. His choice is random, but depends also on the previous day's choice. If on the previous day he bought an egg sandwich, then on the current day he will buy an egg sandwich with probability 0.2 (and a cheese sandwich with probability 0.8). If on the previous day he bought a cheese sandwich, then on the current day he will buy an egg sandwich with probability 0.4 (and a cheese sandwich with probability 0.6).

(a) Model Jack's breakfast with a Markov chain. What are the states? Calculate the transition probability matrix.

(b) Assuming he buys a cheese sandwich today, what is the probability that he will buy cheese sandwich 2 days from now?

(c) A cheese sandwich costs 350 HUF while an egg sandwich costs 250 HUF. Calculate the long term average cost of Jack's breakfast per day.

Solution.

(a) The states are 1: egg sandwich, 2: cheese sandwich. Since Jack's choice depends only on the previous day, the Markov property holds. The transition probability matrix is

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}.$$

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Solution.

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(b) He buys a cheese sandwich today, so

$$v_0 = (\,0\ 1\,).$$

We need to calculate v_2.

$$\begin{aligned} v_1 &= v_0 \cdot P = (\,0.4\ 0.6\,), \\ v_2 &= v_1 \cdot P = (\,0.32\ 0.68\,), \end{aligned}$$

so the probability that he will buy cheese sandwich 2 days from now is 0.68.

Problem 1

Solution.

(c) The MC is irreducible. We compute the stationary distribution $v_{\mathrm{st}} = (x_1\ x_2)$. Then

$$
\begin{aligned}
0.2x_1 + 0.4x_2 &= x_1, \\
0.8x_1 + 0.6x_2 &= x_2, \\
x_1 + x_2 &= 1,
\end{aligned}
$$

from which

$$v_{\mathrm{st}} = (x_1\ x_2) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

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from which

$$v_{\mathrm{st}} = (x_1\ x_2) = \left(\frac{1}{3}\ \frac{2}{3}\right).$$

Then from the ergodic theorem, the long term average cost of Jack's breakfast is

$$\frac{1}{3} \cdot 250 + \frac{2}{3} \cdot 350 = 317$$

HUF per day.

Problem 2

A simple signal processing device processes each incoming signal for a random time with distribution EXP(1) (in seconds). As long as a signal is being processed, the device ignores all other incoming signals. Signals arrive according to a Poisson process with a rate of 4 signals per minute.

(a) Model the behaviour of the device with a continuous time Markov chain. What are the states? Calculate the generator.

(b) As long as the device is processing a signal, its consumption is 8W. As long as it is idle, its consumption is 0.8W. Calculate the long-term average consumption of the device.

(c) What is the ratio of ignored signals in the long run?

Problem 2

Solution.

(a) There are only 2 states: either the device is idle (state 1) or processing a signal (state 2). According to the problem, the processing time has distribution EXP(1). Since signals arrive according to a PPP, the waiting time for the arrival of a signal is also exponentially distributed, and this is a Markov chain. It is irreducible.

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The generator is

$$P = \begin{bmatrix} -1/15 & 1/15 \\ 1 & -1 \end{bmatrix}$$

because 4 signals per minute is equivalent to $1/15$ signals per second.

Problem 2

(b) We compute the stationary distribution $v_{\mathrm{st}} = (x_1\ x_2)$. Then

$$-1/15x_1 + x_2 = 0,$$
$$1/15x_1 - x_2 = 0,$$
$$x_1 + x_2 = 1,$$

from which

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Then from the ergodic theorem, the long-term average consumption of the device is

$$\frac{15}{16} \cdot 0.8 + \frac{1}{16} \cdot 8 = 1.25\ (\mathrm{W}).$$

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(c) The device ignores signals while processing, which is $x_2 = \frac{1}{16}$ of the time, so the ratio of ignored signals is $\frac{1}{16}$.

Problem 3

People are queuing at an ice cream vendor. On average, one person arrives every minute. If he sees at least 2 people in the queue, he leaves immediately; otherwise, he enters the queue. Servicing each person takes 30 seconds on average.

(a) Model the length of the queue with a continuous time Markov chain. What are the states? Calculate the generator.

(b) What is the probability that at a random time, a person is being serviced with nobody else waiting?

(c) A person buys on average 1.8 balls of ice cream. What is the long-term average amount of ice cream balls sold per minute?

Solution.

(a) The states are 0, 1, 2 according to the number of people standing in queue. If the interarrival time and the service time both have exponential distribution, then this is a continuous time Markov chain, notably an M/M/1/2 queue with arrival rate $\lambda = 1$ per minute (because one person arrives every minute on average) and service rate $\mu = 2$ (each person takes 30 seconds to serve on average), and

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}.$$

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The Markov chain is irreducible.

Problem 3

Solution.

(b) We compute the stationary distribution $v_{\mathrm{st}} = (x_0 \, x_1 \, x_2)$.

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$$1 \cdot x_0 = 2 \cdot x_1, \qquad 1 \cdot x_1 = 2 \cdot x_2$$

we have

$$x_0 : x_1 : x_2 = 4 : 2 : 1,$$

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(b) We compute the stationary distribution $v_{\mathrm{st}} = (x_0 \; x_1 \; x_2)$. From the balance equations

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The probability that at a random time, a person is being serviced with nobody else waiting is

$$x_1 = \frac{2}{7}.$$

(c) There is service during states 1 and 2, which is $x_1 + x_2$ of the time; on average, 2 customers per minute is served, and on average, 1.8 balls of ice cream are sold per customer, so according to the ergodic theorem, the long-term average amount of ice cream balls sold per minute is

$$(x_1 + x_2) \cdot 2 \cdot 1.8 = \frac{3}{7} \cdot 2 \cdot 1.8 \approx 1.54.$$

Problem 4

The lifetime of a certain type of lightbulb has exponential distribution with unknown parameter λ. We have a sample of 5 lightbulbs whose lifespan (in hours) is: 629, 48, 450, 4977, 1746. Give a moment estimate for λ.

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Solution. For $X_1 \sim \mathrm{EXP}(\lambda)$,

$$g(\lambda) = \mathbb{E}_\lambda(X_1) = \frac{1}{\lambda},$$

and

$$g^{-1}(x) = \frac{1}{x}.$$

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Then the moment estimate for λ is

$$\hat{\lambda} = g^{-1}(\bar{x}) = g^{-1}\left(\frac{629 + 48 + 450 + 4977 + 1746}{5}\right) = \frac{1}{1570}.$$

Problem 5

When someone takes an IQ test, the result is random, with expectation equal to the person's real IQ, and deviation equal to 3. Jack claims that he has an IQ of 120. He takes 5 separate IQ tests to prove it; his results are 118, 116, 119, 119, 118. Test the hypothesis that Jack has an IQ equal to 120 against the hypothesis that his IQ is not 120 with a confidence level of 95%.

Problem 5

When someone takes an IQ test, the result is random, with expectation equal to the person's real IQ, and deviation equal to 3. Jack claims that he has an IQ of 120. He takes 5 separate IQ tests to prove it; his results are 118, 116, 119, 119, 118. Test the hypothesis that Jack has an IQ equal to 120 against the hypothesis that his IQ is not 120 with a confidence level of 95%.

Solution. The hypotheses are

- H_0: Jack's IQ is $\mu = 120$;
- H_0: Jack's IQ is not equal to 120.

σ is known, so we need to do a one-sample, 2-tail z-test.

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We compute the statistic:

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$$z_{\varepsilon/2} = \Phi^{-1}(0.975) = 1.96.$$

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The comparison

$$z = -1.49 \in [-z_{\varepsilon/2}, z_{\varepsilon/2}] = [-1.96, 1.96]$$

holds, so we accept H_0, that Jack's IQ is 120.

The football team Pegleg FC has 5 strikers. Each striker *that is playing* becomes injured on average once every 4 months. On average, each injury lasts for 2 months. The team always plays with 3 strikers, unless less than 3 are healthy, in which case all healthy strikers play. (So, for example, if all 5 strikers are healthy, 3 of them are playing and the other 2 are not. Only the 3 strikers playing may become injured.)

(a) Model the number of healthy strikers with a CTMC. What are the states? Calculate the generator.

(b) Assuming right now all strikers are healthy, what is the approximate probability that an injury will happen in the next 10 days? (You may assume 1 month has 30 days.)

(c) Calculate the stationary distribution.

(d) What is the probability that they have to play with less than 2 strikers at a random match?

(e) What is the average number of injured strikers over a long period of the time?

(a) The states are 0, 1, 2, 3, 4, 5 according to the number of healthy strikers.

$$Q = \begin{bmatrix} * & 5/2 & 0 & 0 & 0 & 0 \\ 1/4 & * & 4/2 & 0 & 0 & 0 \\ 0 & 2/4 & * & 3/2 & 0 & 0 \\ 0 & 0 & 3/4 & * & 2/2 & 0 \\ 0 & 0 & 0 & 3/4 & * & 1/2 \\ 0 & 0 & 0 & 0 & 3/4 & * \end{bmatrix}$$

Homework 6

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Injuries: if they have 3, 4 or 5 healthy strikers, 3 are playing, so the injury rate to any of them is 3/4 (injuries per month).

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Returning from injury: each injured striker returns with rate 1/2. If there are several strikers injured, any of them may return, so the return rate is proportional to the number of injured strikers.

(b) When all strikers are healthy, then the rate of injury is 3/4, and the waiting time for the next injury is $T \sim \mathrm{EXP}(3/4)$.

$$\mathbb{P}(\text{an injury will happen in the next 10 days}) =$$
$$\mathbb{P}(T < 10) = 1 - e^{-10/30 \cdot 3/4} \approx 0.221.$$

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(c) This is a birth-death process, so the balance equations hold for the stationary distribution:

$$x_0 \cdot 5/2 = x_1 \cdot 1/4, \quad x_1 \cdot 4/2 = x_2 \cdot 2/4, \quad x_2 \cdot 3/2 = x_3 \cdot 3/4,$$
$$x_3 \cdot 2/2 = x4 \cdot 3/4, \quad x_3 \cdot 2/2 = x_4 \cdot 3/4, \quad x_4 \cdot 1/2 = x_5 \cdot 3/4,$$

from which

$$x_0 : x_1 : x_2 : x_3 : x_4 : x_5 = 1 : 10 : 40 : 80 : \frac{320}{3} : \frac{640}{9},$$

and

$$v_{\mathrm{st}} = (0.0032 \;\; 0.0324 \;\; 0.1295 \;\; 0.2591 \;\; 0.3455 \;\; 0.2303).$$

(d) The probability that at a random time they have to play with fewer than 2 strikers is

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(e) According to the ergodic theorem, the average number of injured strikers is

$$x_0 \cdot 5 + x_1 \cdot 4 + \cdots + x_5 \cdot 0 = 1.398.$$